TODIM under Group Decision Making based on Weighted Selective Aggregated Majority-OWA Operator

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ABSTRACT

TODIM (an acronym in Portuguese for Interactive Multiple Criteria Decision Making) is one of the discrete MCDM methods whose fundamental feature is to compare alternatives with respect to each criterion in terms of gains and losses. The inclusion of risk behaviour of expert is one of its key advantages. TODIM was recently extended to deal with group decision making (GDM) problems. However, the collective decision in the existing method is merely based on the aggregation of experts with the main emphasis on full consensus measure. In this article, we propose an extension of TODIM under GDM based on the WSAM-OWA operator which generalizes the aggregation process to include the majority-based consensus measure. Under this operator, the majority and minority concepts can be integrated as to represent a collective decision to deal with the complex decision-making problems. Moreover, it provides the flexibility to significantly analyse the experts' judgments with different degree of importance. A numerical example in strategy investment selection problem was presented to illustrate the proposed method. Based on the analysis, the cardinality relevant factor (CRF) as a measure of partial consensus in TODIM-GDM has shown a significant impact on the final ranking.

Keywords: *Aggregation Operators, Multiple Criteria Decision Making, TODIM, WSAM-OWA.*

I. INTRODUCTION

Nowadays, the analysis of decision-making problems with respect to multi-dimensional aspects has becomes an important focus in scientific research. Multi-criteria decision making (MCDM) is one of these aspects which able to solve the decision-making problems under the multiple conflicting criteria or attributes. The main purpose of this research area is to develop mathematical models to cope with MCDM problems. These methods play a critical role in many real-life applications such as engineering, business, medicine, etc. An increasing number of literatures in MCDM shows this subject is significantly studied and applied in various fields [1-3].

Group decision making (GDM) is another study under the multi-dimensional aspects of decision-making process. It is an analysis of decision-making problems with respect to a group of experts, specifically in a single criterion decision-making environment. In general, there are four preferences or choices to find the solution for the GDM problem, i.e. majority rules, minority rules, individual and unanimity [4].

The past decades have seen a substantial amount of GDM problem-related research [5-7].

In [8], an extension of TODIM under GDM has been proposed. TODIM is a discrete method inspired by the prospect theory [9] and was introduced by Gomes and Lima [10] in the early 1990s. The method is used to determine the best alternative by aggregating all measures of gains and losses over all criteria based on the additive difference function [11]. The main advantage of the method is it takes into account the risk behaviour of experts in making the final decision. Moreover, the best alternative or a subset of alternatives can be selected with respect to the knowledge and opinions of several experts instead of only one expert. There are many studies on TODIM and TODIM-GDM reported in the literature to demonstrate the applicability of the methods. An exhaustive review on the applications and extensions of TODIM method can be referred to [12-14].

In TODIM-GDM methods, the arithmetic mean (AM) and the weighted arithmetic mean (WAM) are among the formally used aggregation operators in generating the representative value of experts as a group decision. However, these operators are merely based on the average of all experts' judgments (i.e., full consensus measure) without any consideration to the soft aggregation processes (i.e., partial consensus measure) that include the majority and/or the minority concept(s). Furthermore, the effect of closeness or similarity between experts' judgments is not explicitly considered in the previous aggregation operators. In this study, the generalized aggregation operator called the weighted selective aggregated majority-ordered weighted average (WSAM-OWA) is proposed to be integrated with TODIM-GDM. This in general to provide a more flexibility in aggregating the experts' judgments with the inclusion of the majority and minority concepts, inclusively. To the best of our knowledge, this is the first study to generalize the TODIM-GDM method with the inclusion of these soft and flexible concepts. The main aim of this study is two-fold: i) to extend the TODIM-GDM under the indirect approach based on the WSAM-OWA operator; and ii) to compare the results of the proposed method with respect to majority, average all, and minority GDM formulation and also the risk behaviour of experts. This extended method provides a greater flexibility to the experts in analysing the GDM result from the majority to the minority GDM perspective. In general, the proposed method not only can model the risk behaviour of experts with respect to gains and losses but also analyse the overall GDM result in accordance with the majority and minority concepts.

The remaining part of this paper is organized as follows: Section 2 provides an overview of the related concepts, definitions and procedure of TODIM method, GDM for the direct and indirect methods, and WSAM-OWA operator. In section 3, the proposed method TODIM-GDM based on WSAM-OWA operator is presented. In section 4, a numerical example is given to illustrate the proposed method. Finally, conclusions are provided in Section 5.

II. PRELIMINARIES

This section reviews some basic concepts and methods related to this study. These include the method of TODIM, two general approaches of GDM method, the ordered weighted averaging (OWA) and also the majority-based aggregation operators.

A. TODIM method

The main idea of TODIM is in measuring the dominance degree of each alternative over the remaining ones, by the pairwise comparison, with respect to the prospect value function. Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a finite set of alternatives and $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be a finite set of criteria with M and N represent the sets of indices, respectively. Assume that the performance of all alternatives with respect to all criteria are known. Denote x_{ij} as the performance value of alternative A_i with respect to criterion C_j. The step-by-step procedure for the TODIM method can be given as the following [10]:

Step 1: Normalize decision matrix $X = [x_{ij}]_{m\times n}$ into $\bar{X} = [\bar{x}_{ij}]_{m\times n}$ using any of the normalization methods (see [15], for some examples). This procedure is to ensure that all the criteria are converted into the same unit or dimension. For the benefit criteria, it is expected that the higher the values the better the criteria, while for the cost criteria, the lower the better.

Step 2: For each $j \in N$, calculate the relative weight of criterion C_j to a reference criterion C_r:

$$w_{jr} = \frac{w_j}{w_r}, \quad (1)$$

where the reference criterion C_r has to be chosen by the expert or basically the highest weight, such that $w_r = max_{k\in N} w_k$.

Step 3: For each $i, h \in M$ and $j \in N$, calculate the dominance degree of alternative A_i over alternative A_h with respect to criterion C_j through the following expression:

$$\Phi_j(A_i, A_h) = \begin{cases} \left(\frac{w_{jr}(\bar{x}_{ij}-\bar{x}_{hj})}{\sum_{k=1}^{n} w_{kr}}\right)^{\frac{1}{2}} & if \quad \bar{x}_{ij} > \bar{x}_{hj} \\ \frac{1}{-\theta}\left(\frac{(\sum_{k=1}^{n} w_{kr})(\bar{x}_{hj}-\bar{x}_{ij})}{w_{jr}}\right)^{\frac{1}{2}} & if \quad \bar{x}_{ij} < \bar{x}_{hj} \end{cases} \quad (2)$$

where $\theta > 0$ is an attenuation factor for the losses (i.e., when $\bar{x}_{ij} < \bar{x}_{hj}$). If $\theta > 1$ the losses are attenuated, while if $\theta < 1$ the losses are amplified. Hence, this parameter is used to incorporate the risk behaviour of expert in ranking the alternatives with respect to gains and losses. Specifically, for large values of θ the best alternatives are those that provide more gains. For small values of θ the best alternatives are those that provide small losses.

Step 4: For each $i, h \in M$, calculate the overall dominance degree of alternative A_i over alternative A_h:

$$\Phi(A_i, A_h) = \sum_{j=1}^{n} \Phi_j(A_i, A_h). \quad (3)$$

Step 5: For each $i \in M$, calculate the overall performance of alternative A_i:

$$\Phi(A_i) = \sum_{h=1}^{m} \Phi(A_i, A_h). \quad (4)$$

Step 6: For each $i \in M$, calculate the normalized overall performance of alternative A_i by using the following expression:

$$\xi(A_i) = \frac{\Phi(A_i) - min_{h\in M}\Phi(A_h)}{max_{h\in M}\Phi(A_h) - min_{h\in M}\Phi(A_h)}. \quad (5)$$

Step 7: Rank alternatives according to the values $\xi(A_i)$. Alternative with the highest $\xi(A_i)$ value shall be chosen as the best alternative.

In [11], the generalized TODIM was proposed to provide a more general parametric form for the risk behaviour function. The subjective value function as introduced by [16] is adopted as follows:

$$\mu(x) = \begin{cases} x^{\alpha} & if\ x \geq 0 \\ -\lambda(-x)^{\beta} & if\ x < 0 \end{cases}, \quad (6)$$

where $\alpha, \beta \in (0,1)$ are coefficients determining the convexity and concavity of the function, respectively. This can be represented by the S-shaped value function, which are concave in the first quadrant (representing the risk aversion over gain) and convex in the third quadrant (representing the risk-seeking over loss) [17]. The parameter $\lambda > 1$, such that $\lambda = 1/\theta$, is called the loss aversion coefficient. Then, the new function of dominance degree of alternative A_i over alternative A_h can be defined as:

$$\Phi_j(A_i, A_h) = \begin{cases} (w_j(\bar{x}_{ij} - \bar{x}_{hj}))^{\alpha} & if\ \ \bar{x}_{ij} \geq \bar{x}_{hj} \\ -\lambda\left(\frac{(\bar{x}_{hj} - \bar{x}_{ij})}{w_j}\right)^{\alpha} & if\ \ \bar{x}_{ij} < \bar{x}_{hj} \end{cases}, \quad (7)$$

where the relative weight of each criterion is simplified as [17]:

$$\frac{w_{jr}}{\sum_{k=1}^{n} w_{kr}} = \frac{w_j(w_r)^{-1}}{\sum_{j=1}^{n} w_k(w_r)^{-1}} = \frac{w_j}{\sum_{k=1}^{n} w_k} = w_j. \quad (8)$$

In this generalized model, β is set to be equal to α.

B. Group decision-making method

GDM method is a collective decision of experts. Note that, the aforementioned TODIM method is based on a judgment of a single expert. To deal with TODIM under GDM, basically, there are two general frameworks available in the literature. In general, it can be classified as direct (classical scheme) and indirect (alternative scheme) approaches [18].

- Direct approach: Obtained without constructing a social (collective) opinion on decision matrices of experts. But the collective decision is based on the vectors of individual priority/ranking of experts.

Indirect approach: A social opinion with respect to the aggregation of decision matrices of experts is conducted to solve the GDM problem. Specifically, this approach is based on the aggregation of experts' judgments with respect to each criterion instead on each alternative.

Generally, there are three elements to characterize the GDM problems [18]:

- The existence of decision-making problem with specific objective to achieve.
- A finite set $\mathcal{A}$ of $m \geq 2$ alternatives to the problem: $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$.
- A group $\mathcal{E}$ of $p \geq 2$ experts to evaluate the problem: $\mathcal{E} = \{E_1, E_2, \dots, E_p\}$.

C. Majority Additive-OWA operators

OWA operator is a parameterized family of mean type aggregation operators [19]. It provides a flexibility to aggregate the argument values between two extreme cases, i.e., minimum to maximum. In [20], the MA-OWA which is a new family of OWA (i.e., neat-OWA) has been proposed. The definition of MA-OWA can be given as follows.

Definition 1 [20]: A MA-OWA is a function $\mathcal{F}_{MA}: \mathbb{R}^p \times \mathbb{N}^{\mathcal{N}} \to \mathbb{R}$ defined as:

$$\mathcal{F}_{MA}(x_1, x_2, \dots, x_p) = \sum_{s=1}^{p} w_{s,\mathcal{N}} b_{\sigma(s)} \quad (9)$$

where $\mathcal{N} = \max_{1 \leq s \leq p} m_s$ and $\sigma(s)$ denotes a permutation of group of argument b_s with respect to cardinality m_s, such that $b_{\sigma(s)} \geq b_{\sigma(s+1)}$.

There are some other families of MA-OWA which can be summarized as [21]:

- Selective MA-OWA (SMA-OWA): This operator generalizes the MA-OWA by introducing the cardinality relevant factor (CRF), δ, such that $\delta \in [0,1]$. The initial weight is set as equally important for each group of cardinalities of argument values, $w_{s,1} = 1/p$, in the similar way as in the case of MA-OWA.
- Selective Aggregated Majority-OWA (SAM-OWA): The CRF is used as above, however, the initial weights are based on the average of their cardinalities, $w_{s,1} = m_s / \sum_{t=1}^{p} m_t$.
- Weighted Selective Aggregated Majority-OWA (WSAM-OWA): The initial weights are denoted as $w_{s,1} = \sum_{s=1}^{m_s} v_s$. This operator generalizes the other family of MA-OWA operators where it can be reduced to the SAM-OWA when $v_s = m_s / \sum_{t=1}^{p} m_t$ and for $v_s = 1/p$, the SMA-OWA can be recovered. In specific, the definition of WSAM-OWA can be given as follows:

Definition 2 [21]: A WSAM-OWA operator is a function $\mathcal{F}_{WSAM}: \mathbb{R}^p \times \mathbb{N}^{\mathcal{N}} \to \mathbb{R}$ that has an associated weighting vector V of dimension p such that $\sum_{s=1}^{p} v_s = 1$ and $v_s \in [0,1]$, is defined as:

$$\mathcal{F}_{WSAM}(x_1, x_2, \dots, x_p) = \sum_{s=1}^{p} w_{s,\mathcal{N}} b_{\sigma(s)}, \quad (10)$$

where $\mathcal{N} = \max_{1 \leq s \leq p} m_s$, and σ denotes a permutation with respect to the cardinality m_s. The associated weights are defined by the recurrent relations:

$$w_{s,1} = \omega_s = \begin{cases} v_s, & if \ \ m_s = 1, \\ \sum_{s=1}^{m_s} v_s, & if \ \ m_s > 1, \end{cases} \quad (11)$$

and the cardinal-dependent weights are given as:

$$w_{s,u} = \frac{\omega_s \gamma_{s,u} y_u + w_{s,u-1}}{z_u}, \quad (12)$$

$$y_1 = 1, y_u = \begin{cases} 1, & if \ \sum_{t=1}^{p} \omega_t \gamma_{t,u} = 0, \\ \frac{\sum_{t=1}^{p} \gamma_{t,u}}{\sum_{t=1}^{p} \omega_t \gamma_{t,u}}, & otherwise, \end{cases} \quad (13)$$

$$z_1 = 1, z_u = \begin{cases} 1, & if \ \sum_{t=1}^{p} \omega_t \gamma_{t,u} = 0, \\ 1 + \sum_{t=1}^{p} \gamma_{t,u}, & otherwise, \end{cases} \quad (14)$$

where $\gamma_{s,u}$ is given as:

$$\gamma_{s,u} = \begin{cases} \delta & m_{\sigma(s)} \geq u, \\ 1 - \delta & otherwise. \end{cases} \quad (15)$$

The parameter δ is the CRF with $\delta \in [0\ 1]$ and $1 \leq s \leq p$, $2 \leq u \leq \mathcal{N}$. Note that u factor represents the current cardinality considered at a moment in the aggregation process.

Example 1 shows a simple computation of WSAM-OWA with $v_s = m_s / \sum_{t=1}^{p} m_t$, or in the case of $\mathcal{F}_{WSAM} = \mathcal{F}_{SAM}$. Consider that $\mathcal{E}$ = {0.3, 0.3. 0.3, 0.3, 0.3, 0.3, 0.3, 0.4, 0.4, 0.4, 0.6, 0.6, 0.6, 0.6, 0.7} is a set or argument values and the weight associated to each argument is set as equally important. Formally, it can be represented as x_s = {(0.3, 7, 0.467), (0.4, 3, 0.200), (0.6, 4, 0.267), (0.7, 1, 0.067)}, where $x_s = (b_s, m_s, \omega_s)$. Table 1 depicts a simulation result for $\delta = 1$, the tendency towards the majority group.

Table 1. Value of $\gamma_{s,u}$, u_u and y_u.

	v_1 0.7 m_1	v_2 0.4 m_2	v_3 0.6 m_3	v_4 0.3 m_4	$\delta = 1$ z_u	y_u
$\gamma_{s,u}$	1	3	4	7	15	
$\gamma_{s,1}$	1.00	1.00	1.00	1.00	4.00	1.00
$\gamma_{s,2}$	0.00	1.00	1.00	1.00	4.00	3.21
$\gamma_{s,3}$	0.00	1.00	1.00	1.00	4.00	3.05
$\gamma_{s,4}$	0.00	0.00	1.00	1.00	3.00	2.56
$\gamma_{s,5}$	0.00	0.00	0.00	1.00	2.00	1.69
$\gamma_{s,6}$	0.00	0.00	0.00	1.00	2.00	1.26
$\gamma_{s,7}$	0.00	0.00	0.00	1.00	2.00	1.11

The cardinal-dependent weights are given as:

$w_{1,4} = 0.000, w_{2,4} = 0.009, w_{3,4} = 0.042, w_{4,4} = 0.949,$

and the WSAM-OWA operator yield:

$$\mathcal{F}_{WSAM} = \{(0.3,7,0.467),(0.6,4,0.267),(0.4,3,0.200),(0.7,1,0.067)\} = 0.318.$$

As can be seen, this result, 0.318 is a representative value of the majority of argument values, which is closer to 0.3 with cardinality of 7. In comparison, if we set $\delta = 0$, the minority of argument values is derived, which is 0.679 (i.e., closer to 0.7 with cardinality of 1). While for $\delta = 0.5$, the aggregated value 0.427 is generated. Note that, $\mathcal{F}_{WSAM}$ is reduced to $\mathcal{F}_{WAM}$ whenever $\delta = 0.5$.

III. THE PROPOSED METHOD

In this section, the extension of TODIM under GDM method based on WSAM-OWA operator is presented. The main objective of the proposed method is to integrate the risk behaviour modelling (as in TODIM-GDM method) with the majority concept (in WSAM-OWA operator). In such a way, we can have a greater flexibility in analysing the collective results (i.e., including partial consensus measures) instead of only full consensus measure such in the classical method. All the notations are explained as the following. Recall that a set of alternatives and a set of criteria are presented as $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$, respectively. A weighting vector of criteria is denoted as $W = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Moreover, a group of experts is given as $\mathcal{E} = \{E_1, E_2, \dots, E_p\}$ with $V = (v_1, v_2, \dots, v_p)$ is a weighting vector of experts such that $v_s \in [0,1]$ and $\sum_{s=1}^{p} v_s$=1. Note that, in the indirect GDM approach, the weighting vector of experts can be extended to each specific criterion $j \in N$ such that $V(j) = (v_1^j, v_2^j, \dots, v_p^j)$. This weighting vector specifies the expertise of experts on each criterion under the consideration. Step-by-step procedure is as follows:

Step 1: Each expert, E_s provides his/her preferences for all alternatives, A_i with respect to all criteria, C_j in the form of a decision matrix $X^s = [x_{ij}^s]_{m\times n}$, $s = 1,2,\dots,p$, such that:

$$X^s = [x_{ij}^s]_{m\times n} = \begin{bmatrix} x_{11}^s & \cdots & x_{1n}^s \\ \vdots & \ddots & \vdots \\ x_{m1}^s & \cdots & x_{mn}^s \end{bmatrix}. \quad (16)$$

Step 2: At this stage, all the decision matrices of experts will be aggregated to form a group decision matrix. By integrating the TODIM model with the WSAM-OWA such in equations (10) – (15), a majority-decision matrix, $X^{maj} = [x_{ij}^{maj}]_{m\times n}$ can be formed by aggregating the preferences of all experts, x_{ij}^s with respect to each criterion $j \in M$, given that their associated weights, v_s^j. Note that, by specifying different values for CRF, $\delta \in [0,1]$, various strategies of collective group decision can be formed, namely:

- Majority ($\delta \to 1$) – tend towards the majority or the most similar opinion of experts,
- average all ($\delta = 0.5$) – tend towards the average of all opinion of experts (i.e., the case of classical model),
- minority ($\delta \to 0$) – tend towards the minority opinion of experts.

Moreover, the CRF can also be used as a mediating parameter to avoid the exclusion of the opinion of experts due to a huge different in cardinality between them. Hence, all the experts' judgments can be included in the GDM.

Step 3: Afterwards, normalize X^{maj} to get a new decision matrix, $\bar{X}^{maj} = [\bar{x}_{ij}^{maj}]_{m\times n}$ with respect to benefit and/or cost criteria [15].

Step 4: Next, by specifying any value for λ, which is the parameter of risk behaviour of experts, the dominance degree of alternative A_i over each alternative A_h with respect to criterion C_j can be computed using the following expression:

$$\Phi_j^{maj}(A_i, A_h) = \begin{cases} (w_j(\bar{x}_{ij}^{maj} - \bar{x}_{hj}^{maj}))^{\alpha} & if\ \bar{x}_{ij} \geq \bar{x}_{hj} \\ -\lambda\left(\dfrac{(\bar{x}_{hj}^{maj} - \bar{x}_{ij}^{maj})}{w_j}\right)^{\alpha} & if\ \bar{x}_{ij} < \bar{x}_{hj} \end{cases} \quad (17)$$

where α represents estimated coefficient for the convexity/concavity of the function and $\lambda = 1/\theta$ which are λ <1 (risk averse), λ = 1 (neutral) and λ > 1 (risk taker).

Step 5: For each $i, h \in M$, calculate the overall dominance degree of alternative A_i over alternative A_h using the equations (3) and (4).

Step 6: Rank the alternatives and select the best one(s) according to the overall values of alternatives. The higher the value of $\xi(A_i)$, the better the alternative will be.

IV. NUMERICAL EXAMPLE

In this section, a numerical example on selecting the best investment strategy is given to show the applicability of the proposed method.

A. Case Study: Investment Selection Strategy

The objective of this case study is to guide an investor in analysing and selecting the best investment strategy or alternative with respect to the judgment provided by a group of experts. The alternatives which considered here consists of $A_1 =$ hedge funds, $A_2 =$ investment funds, $A_3 =$ bonds, $A_4 =$ stocks and $A_5 =$ equity derivatives. The main focus here is on the majority opinion of experts that provide similar preferences or based on the highest reliability of experts as the main sources of analysis. Moreover, by specifying certain degree of risk behaviour, a vast spectrum of analysis can be made to guide a decision-making process. To exemplify this analysis, the hypothetical data in [19] is used. Five experts are considered to evaluate the alternatives with respect to five criteria: $C_1 =$ benefits in the short term, $C_2 =$ benefits in the long term, $C_3 =$ risk of the investment, $C_4 =$ social responsible investment and $C_5 =$ difficulty of the investment. In addition, the weighting vector for the criteria is given as $W =$ (0.206, 0.196, 0.206, 0.196, 0.196). The related data are presented in Table 2. These data are provided by experts based on the following linguistic scale: $S = \{s_0 = none,\ s_1 = very\ low,\ s_2 = low,\ s_3 = medium,\ s_4 = high,\ s_5 = very\ high, s_6 = perfect\}$. Then, for the purpose of this study, these data are converted into the crisp values using the transformation function such in [21].

Table 2. The decision matrix of each expert.

	E_1					E_2				
	C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5
A_1	s_3	s_2	s_3	s_2	s_5	s_2	s_5	s_6	s_5	s_5
A_2	s_4	s_6	s_1	s_6	s_2	s_6	s_3	s_1	s_6	s_4
A_3	s_2	s_3	s_2	s_2	s_1	s_1	s_5	s_4	s_3	s_2
A_4	s_5	s_2	s_4	s_6	s_5	s_5	s_1	s_3	s_6	s_5
A_5	s_1	s_6	s_3	s_4	s_5	s_3	s_3	s_5	s_5	s_5

	E_3					E_4				
	C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5
A_1	s_1	s_3	s_5	s_4	s_5	s_1	s_3	s_5	s_4	s_4
A_2	s_5	s_5	s_1	s_6	s_3	s_5	s_3	s_2	s_5	s_2
A_3	s_4	s_4	s_3	s_3	s_2	s_2	s_2	s_1	s_4	s_1
A_4	s_5	s_1	s_4	s_6	s_3	s_3	s_1	s_3	s_3	s_5
A_5	s_4	s_3	s_4	s_5	s_4	s_2	s_2	s_3	s_4	s_5

	E_5				
	C_1	C_2	C_3	C_4	C_5
A_1	s_1	s_2	s_3	s_2	s_4
A_2	s_5	s_4	s_1	s_5	s_1
A_3	s_2	s_2	s_1	s_4	s_3
A_4	s_4	s_4	s_2	s_5	s_4
A_5	s_1	s_2	s_4	s_2	s_4

Step 1: The decision matrix of dimension $m \times n$ for each expert is given in Table 2. Similarly, the weight or reliability of each expert with respect to each criterion is provided in Table 3.

Table 3. The reliability of experts.

	E_1	E_2	E_3	E_4	E_5
C_1	0.83	0.67	0.83	0.5	0.5
C_2	0.67	0.83	0.5	0.67	0.67
C_3	0.5	0.5	0.83	0.67	0.83
C_4	0.67	0.67	0.83	0.67	0.5
C_5	0.5	0.67	0.67	0.83	0.67

Step 2: Determine the collective decision matrix by using the equations (10) – (15). At this stage, the majority-decision matrix is derived by aggregating all the individual decision matrix of experts.
For example, consider the argument values provided by the experts for alternative A_1 with respect to criterion C_1 as follows: $x_{11}^s = \{0.5, 0.33, 0.17, 0.17, 0.17\}$. Given the reliability degree of each expert as in Table 3, the normalized weights of criterion C_1 can be derived as $v_1^s = \{0.25, 0.20, 0.25, 0.15, 0.15\}$, such that $v_1^s \in [0,1]$ and $\sum_{s=1}^{5} v_1^s = 1$. To simplify the notations, we denote $x_{11}^s = x_s$ and $v_1^s = v_s$. Note that $x_s = (b_s, m_s, v_s)$ can be represented as $\{(0.5,1,0.206), (0.33,1, 0.196), (0.17,3,0.598)\}$. Based on these values, the aggregated value that represent the majority opinion of experts, x_{11}^{maj} or x_{maj} for alternative A_1 and criteria C_1 can be determined and the detailed computation is shown in Table 4.

Table 4. WSAM-OWA aggregation for C_1.

	v_1 0.5 m_1	v_2 0.33 m_2	v_3 0.17 m_3	$\delta = 1$	
$\gamma_{s,u}$	1	1	3	z_u	y_u
$\gamma_{s,1}$	1	1	1	3	1.00
$\gamma_{s,2}$	0	0	1	2	1.82
$\gamma_{s,3}$	0	0	1	2	1.29

Then, the cardinality-dependent weights can be derived as:

$$w_{1,3} = 0.062, w_{2,3} = 0.050 \text{ and } w_{3,3} = 0.887$$

the WSAM-OWA operator yields:

$$x_{11}^{maj} = \mathcal{F}_{WSAM}\{(0.5,1,0.206), (0.33,1,0.196), (0.17,3,0.598)\} = 0.196$$

for alternative A_1 and criterion C_1. Similar procedure is implemented to aggregate all the x_{ij}^s in forming the group or collective decision matrix $x^{maj} = [x_{ij}^{maj}]_{m \times n}$. The collective decision matrix of majority of experts is given in Table 5.

Table 5. The collective decision matrix of majority of experts.

	C_1	C_2	C_3	C_4	C_5
A_1	0.196	0.446	0.651	0.542	0.796
A_2	0.504	0.592	0.171	0.971	0.367
A_3	0.346	0.437	0.259	0.557	0.267
A_4	0.814	0.442	0.568	0.969	0.808
A_5	0.275	0.467	0.610	0.734	0.800

Step 3: Normalize the overall aggregated results of majority of experts by using the benefit and cost criteria. In this case, C_1, C_2 and C_4 are classified as the benefit criteria, while C_3 and C_5 are set as the cost criteria. The normalized

decision matrix $\tilde{X}^{maj}$ is presented in the Table 6.

Table 6. The normalized alternatives against criteria.

	C_1	C_2	C_3	C_4	C_5
A_1	0.000	0.057	0.000	0.000	0.023
A_2	0.498	1.000	1.000	0.804	0.816
A_3	0.242	0.000	0.818	0.029	1.000
A_4	1.000	0.029	0.173	0.800	0.000
A_5	0.128	0.190	0.085	0.360	0.015

Step 4: Calculate the partial dominance of each alternative A_i over each alternative A_h, $\Phi_i^{maj}(A_i, A_h)$, $i, h = 1,2, \dots, 5$ using equation (16). Here, the value of λ is set equal to 1 and α is 0.5. Then, calculate the overall dominance degree of matrix. The results are provided in Table 7.

Table 7. The overall degree of dominance.

$\Phi^{maj}_{overall}$					
A_1	0.00	2.00	0.61	0.32	0.52
A_2	-9.99	0.00	-6.11	-6.09	-9.01
A_3	-5.59	0.29	0.00	-3.17	-4.42
A_4	-5.00	-0.28	-3.48	0.00	-3.98
A_5	-3.57	1.80	-1.30	-0.33	0.00

Step 5: Finally, the overall value of each alternative can be obtained using the equation (4).

Step 6: Rank the alternative based on descending order. In this case we have A_1=0.000, A_2=1.000, A_3=0.484, A_4= 0.545 and A_5= 0.266. The final ranking of all alternatives is given as $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$, where the investment fund is ranked as the best strategy for this investment.

V. RESULTS AND DISCUSSION

The finding in the previous section is mainly based on the risk neutral of majority of experts and the curvature of function specified by the values, λ =1, $\delta = 1$ and $\alpha = 0.5$, respectively. In this section, an analysis based on different values of λ, δ and α is conducted to see the effect of these parameters on the final results and rankings of TODIM-GDM method. The overall values and rankings of each alternative are provided in Tables 8, 9 and 10.

Table 8. The overall result for different values of δ when $\alpha = 0.5, \lambda = 1$

δ	**0.01**	**0.2**	**0.5**	**0.7**	**1**
A_1	0.426	0.164	0.000	0.000	0.000
A_2	1.000	1.000	1.000	1.000	1.000
A_3	0.409	0.436	0.487	0.520	0.484
A_4	0.379	0.340	0.378	0.395	0.545
A_5	0.000	0.000	0.049	0.124	0.266

Table 9. The overall result for different values of α when $\lambda = 0.5, \delta = 1$.

α	**0.01**	**0.2**	**0.5**	**0.7**	**1**
A_1	0.000	0.000	0.000	0.000	0.000
A_2	1.000	1.000	1.000	1.000	1.000
A_3	0.431	0.460	0.484	0.497	0.511
A_4	0.434	0.500	0.545	0.552	0.549
A_5	0.286	0.276	0.266	0.268	0.280

Table 10. The overall result for different values of λ when α =0.5, $\delta = 1$.

λ	0.1	0.5	1	5	10
A_1	0.000	0.000	0.000	0.000	0.000
A_2	1.000	1.000	1.000	1.000	1.000
A_3	0.469	0.480	0.484	0.489	0.490
A_4	0.497	0.532	0.545	0.560	0.562
A_5	0.216	0.253	0.266	0.282	0.284

In Table 8, the best strategy for investment is given as A_2 for all values of δ, even though the rankings for $\delta \geq 0.5$ are slightly different. Moreover, by using the median values based on the experimental studies carried out in [21], specifically for $\alpha = 0.88$, $\lambda = 2.25$ and $\delta = 1$, we still get A_2 as the best strategy for investment. Analogously, with respect to α and λ such in Tables 9 and 10, A_2 is consistently ranked as the best option. Hence, we can conclude that the overall results of this case study are not sensitive due the best strategy/alternative, but, slightly sensitive regards to the overall rankings. Therefore, if more than one alternative (or a subset of alternatives) needs to be selected, then, a further analysis must be conducted since some of the rankings are slightly distinct, especially with respect to CRF, δ.

Based on the decision analysis model proposed in this study, there are three cases that might be considered by the investor to guide his/her decision. These cases can be explained by the relation between the proportions of experts with respect to δ and their degrees of reliability (or weight of experts):

Case 1: If for each criterion, all the most reliable experts, v_g included in the majority group $\delta \geq 0.5$, then the majority concept can be considered as the best option.

Case 2: If for each criterion, all the most reliable experts, v_g included in the minority group $\delta < 0.5$, then the minority group or majority group might be considered (based on the tendency of investor). Or further analysis may be examined to see the effect of this tendency towards the final result and ranking.

Case 3: If for each criterion, all the most reliable experts, v_g are scattered between the minority and the majority groups, then the intermediate value may be considered.

To sum up, the proposed method provides an exhaustive analysis of the compensation between the majority group or minority group and also the degree of reliability of experts. Moreover, an extensive analysis on the risk behaviour λ and convexity/concavity function, α might produce a good measure for the final decision analysis.

VI. CONCLUSION

This paper proposes an extension of TODIM under GDM based on the WSAM-OWA operator. The aggregation process in the existing TODIM-GDM model has been extended to include the majority-based consensus measure. Under this operator, the majority concept (as well as the minority ones) can be integrated as to represent a collective decision to deal with complex decision-making problems. The advantage of the proposed method is that the vast analysis of heterogeneous GDM problems can be conducted with respect to not only the risk behaviour of experts, but also with certain proportion of experts (from minority to majority) with different degrees of reliability. To exemplify the applicability of the proposed method, a numerical example in investment selection problem is presented. For further research, an extension of TODIM-GDM with WSAM-OWA for the direct approach will be conducted. Then a comparison between direct and indirect approaches of TODIM-GDM with WSAM-OWA operator will be studied. Moreover, the model can be further generalized to include the attitudinal character of experts (i.e., either towards optimistic or pessimistic) with respect to criteria in the selection process.

ACKNOWLEDGEMENT

This work was partly supported by Universiti Malaysia Terengganu (UMT) internal research grant (TAPE-RG), reference no: 55109. The authors are grateful to anonymous reviewers for their comments. Special thanks to Associate Prof. Dr. Abdul Fatah Wahab and Prof. Dr. Abu Osman Md Tap for their continuous support and encouragement.

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